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Boundary Hemivariational Inequalities of Hyperbolic Type and Applications*

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Abstract. In this paper we examine two classes of nonlinear hyperbolic initial boundary value problems with nonmonotone multivalued boundary conditions characterized by the Clarke subdifferential. We prove two existence results for multidimensional hemivariational inequalities: one for the inequalities with relation between reaction and velocity and the other for the expressions containing the reaction–displacement law. The existence of weak solutions is established by using a surjectivity result for pseudomonotone operators and a priori estimates. We present also an example of dynamic viscoelastic contact problem in mechanics which illustrate the applicability of our results.

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1. Introduction

In this paper we consider dynamic hemivariational inequalities involving nonmonotone and possibly multivalued relations as boundary conditions. Such inequalities result from the d'Alembert principle for dynamic mechanical systems. The notion of hemivariational inequality was introduced by Panagiotopoulos in the early eighties as variational formulation of mechanical problems with the nonsmooth and nonconvex energy functionals. This formulation is based on the notion of the Clarke subdifferential for locally Lipschitz functions. The mathematical results on the stationary hemivariational inequalities can be found in Panagiotopoulos [30,31,34], Naniewicz and Panagiotopoulos [28], Motreanu and Panagiotopoulos [27], Haslinger et al. [14] and the references therein. As concerns the parabolic hemivariational inequalities, we refer to Carl [5,6], Liu [16], Miettinen [17], Miettinen and Panagiotopoulos [18,19], Migorski [21–23] and Migorski and Ochal [26]. We mention that the study of hyperbolic hemivariational

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inequalities was initiated by Panagiotopoulos [32–34] who considered models characterized by one-dimensional reaction–velocity laws. The hyperbolic hemivariational problems with a subdifferential relation depending on the first order derivative were considered by Goeleven et al. [12], Haslinger et al. [14] and Gasinski [10] while the dynamic inequalities with a multivalued term depending on the unknown function were treated by Panagiotopoulos and Pop [35], Haslinger et al. [14], Xingming [37] who used the Galerkin method and by Gasinski and Smolka [11], Goeleven and Motreanu [13] (one dimensional wave equation), Migorski [22] and Ochal [29].

The optimal control problems for dynamical hemivariational inequalities have been studied only recently (cf. Gasinski [10], Migorski [20, 22], Migorski and Ochal [25] and Ochal [29]). For a review of other results on hemivariational inequalities cf. Migorski [24].

In the present paper we treat two types of dynamic hemivariational inequalities of hyperbolic type with the subdifferential boundary conditions. The considered cases are following: the hemivariational inequality with the multivalued relations between reaction and displacement and the hemivariational inequality containing the multivalued reaction–velocity laws. More precisely the first type of hemivariational inequalities studied in this paper is of the form:

$$\begin{cases} \langle u''(t)+A(t,u'(t))+Bu(t)-f(t),v\rangle_{V^*\times V}+ \\ +\int_\Gamma j^0(x,t,\gamma u(t);\gamma v)\mathrm{d}\sigma(x)\geqslant 0 \text{ for all } v\in V \text{and a.e. } t\in(0,T) \\ u(0)=u_0, \quad u'(0)=u_1, \end{cases} \qquad (P)$$

where $A(t,\cdot)\colon V\to V^*$ is a nonlinear operator, B is a linear bounded operator from V into its dual V^*, $f\in\mathcal{V}^*$, $u_0\in V$, $u_1\in H$, Γ is a regular part of the boundary of an open bounded subset Ω of $\mathbb{R}^N$, $j^0(x,t,u;v)$ is the Clarke directional derivative of a locally Lipschitz function $j(x,t,\cdot)\colon\mathbb{R}^N\to\mathbb{R}$ at point $u\in\mathbb{R}^N$ and the direction $v\in\mathbb{R}^N$, $\gamma\colon H^1(\Omega;\mathbb{R}^N)\to L^2(\partial\Omega;\mathbb{R}^N)$, $\mathcal{V}=L^2(0,T;V)$, V is a closed subspace of $H^1(\Omega;\mathbb{R}^N)$, $H=L^2(\Omega;\mathbb{R}^N)$ and $V\subset H\subset V^*$ form of an evolution triple of spaces. In the second type of hemivariational inequalities we work with a locally Lipschitz function which depends on the first order time derivative of u.

We would like to underline that our existence results (Theorems 6 and 11) are applicable to hemivariational inequalities with multidimensional superpotential laws, i.e., for hemivariational inequalities on vector-valued function spaces. The idea of the proofs of Theorems 6 and 11 consists of two steps. Firstly, we assume a regular data $u_1\in V$ and we formulate the problem (P) as an evolution inclusion of first order. Next we use the surjectivity result (see Theorem 2.1 of [36]) and we get the existence of solutions of the first order problem. In the second step we remove the restriction on the initial datum and we prove the result for $u_1\in H$.

Moreover, in contrast to [10,12,14], we consider the nonlinear damping term which satisfies a general pseudomonotonicity condition, we do not assume the coercivity of the linear operator B and we suppose less restrictive hypotheses on a locally Lipschitz function generating the subdifferential relation. In this way we avoid several restictive assumptions needed in the above mentioned papers.

The paper is organized as follows. After the preliminary material of Section 2, in Section 3 we introduce two classes of boundary hemivariational inequalities and their formulations as operator evolution inclusions. The main existence theorems with the crucial lemmas are established in Section 4. The proofs of auxiliary results are provided in Section 5. Finally, in the last section we consider the applications to contact problems in mechanics.

2. Preliminaries

In this section we fix the notation and recall some definitions needed in the sequel. Given a reflexive Banach space Y, we denote by $\langle \cdot, \cdot \rangle_Y$ the pairing between Y and its dual Y^*. We recall some definitions for a multivalued operator $T: Y \to 2^{Y^*}$ (see e.g. Browder and Hess [4] and Zeidler [38]).

An operator T is said to be *pseudomonotone* if it satisfies

(a) for every $y \in Y$, Ty is a nonempty, convex and weakly compact set in Y^*;
(b) T is u.s.c. from every finite dimensional subspace of Y into Y^* endowed with the weak topology; and
(c) if $y_n \to y$ weakly in Y, $y_n^* \in Ty_n$ and $\limsup \langle y_n^*, y_n - y \rangle_Y \leqslant 0$, then for each $z \in Y$ there exists $y^*(z) \in Ty$ such that $\langle y^*(z), y - z \rangle_Y \leqslant \liminf \langle y_n^*, y_n - z \rangle_Y$.

Let $L: D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator. An operator T is said to be *pseudomonotone with respect to* $D(L)$ (shortly L *pseudomonotone*) if and only if (a) and (b) hold and

(d) if $\{y_n\} \subset D(L)$ is such that $y_n \to y$ weakly in Y, $Ly_n \to Ly$ weakly in Y^*, $y_n^* \in T(y_n)$, $y_n^* \to y^*$ weakly in Y^* and $\limsup \langle y_n^*, y_n \rangle_Y \leqslant \langle y^*, y \rangle_Y$, then $(y, y^*) \in \text{Graph}(T)$ and $\langle y_n^*, y_n \rangle_Y \to \langle y^*, y \rangle_Y$.

An operator T is said to be *coercive* if there exists a function $c: \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$ such that $\langle y^*, y \rangle_Y \geqslant c(||y||_Y)\, ||y||_Y$ for every $(y, y^*) \in \text{Graph}(T)$. A single-valued operator $T: Y \to Y^*$ is said to be *demicontinuous* if it is continuous from Y to Y^* endowed with weak topology. $T: Y \to Y^*$ is *pseudomonotone* if for each sequence $\{y_n\} \subseteq Y$ such that it converges weakly to $2y_0 \in Y$ and $\limsup \langle Ty_n, y_n - y_0 \rangle_Y \leqslant 0$, we have $\langle Ty_0, y_0 - y \rangle_Y \leqslant \liminf \langle Ty_n, y_n - y \rangle_Y$ for all $y \in Y$.

The following surjectivity result (see Papageorgiou, Papalini and Renzacci [36]) for L pseudomonotone operators will be used in our existence theorems.

PROPOSITION 1. *If Y is a reflexive, strictly convex Banach space, $L\colon D(L) \subset Y \to Y^*$ is a linear densely defined maximal monotone operator and $T\colon Y \to 2^{Y^*} \backslash \{\emptyset\}$ is bounded coercive and pseudomonotone with respect to $D(L)$, then $L + T$ is surjective.*

Finally, we recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $h\colon E \to \mathbb{R}$, where E is a Banach space (see Clarke [8]). The generalized directional derivative of h at x in the direction v, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{h(y + tv) - h(y)}{t}.$$

The generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space E^* given by $\partial h(x) = \{\zeta \in E^*\colon h^0(x; v) \geqslant \langle \zeta, v \rangle_{E^* \times E}$ for all $v \in E\}$. The locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

3. Problem Formulation

In this section we state the hypotheses on the data of the problems and we present some auxiliary material that will be used in the proofs of our main results in Section 4.

Let Ω be an open bounded subset of $\mathbb{R}^N$ with a Lipschitzean boundary $\partial\Omega$ and let Γ be an open subset of $\partial\Omega$ with positive surface measure. Let $V = \{v \in H^1(\Omega; \mathbb{R}^N) : \gamma v = 0$ on $\partial\Omega \setminus \Gamma\}$ and $H = L^2(\Omega; \mathbb{R}^N)$, where $\gamma\colon H^1(\Omega; \mathbb{R}^N) \to L^2(\partial\Omega; \mathbb{R}^N)$ denotes the trace operator. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ (see Lions [15], Zeidler [38]) with dense, continuous and compact embeddings. We denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality of V and its dual V^* as well as the inner product on H, by $||\cdot||$, $|\cdot|$ and $||\cdot||_{V^*}$ the norms in V, H and V^*, respectively. In what follows we need the spaces $\mathcal{V} = L^2(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$ and $\mathcal{W} = \{w \in \mathcal{V}\colon w' \in \mathcal{V}^*\}$, where the time derivative involved in the definition of $\mathcal{W}$ is understood in the sense of vector valued distributions. Equipped with the norm $||v||_{\mathcal{W}} = ||v||_{\mathcal{V}} + ||v'||_{\mathcal{V}^*}$ the space $\mathcal{W}$ becomes a separable reflexive Banach space. We also have $\mathcal{W}\subset\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}^*$. The duality for the pair $(\mathcal{V},\mathcal{V}^*)$ is denoted by $\langle\langle f,v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}=\int_0^T\langle f(t),v(t)\rangle\,\mathrm{d}t$. It is well known (cf. Lions [15] and Zeidler [38]) that the embeddings $\mathcal{W}\subset C(0,T;H)$ and $\{w\in\mathcal{V}:\ w'\in\mathcal{W}\}\subset C(0,T;V)$ are continuous.

We consider the following two types of dynamic hemivariational inequalities.

PROBLEM (I). Find $u\in\mathcal{V}$ such that $u'\in\mathcal{W}$ and

$$\begin{cases}\langle u''(t)+A(t,u'(t))+Bu(t)-f(t),v\rangle_{V^*\times V}\\ \quad+\int_\Gamma j^0(x,t,\gamma u(t);\gamma v)\,\mathrm{d}\sigma(x)\geqslant 0 \quad \text{for all } v\in V \text{ and a.e. } t\in(0,T)\\ u(0)=u_0,\ \ u'(0)=u_1,\end{cases}$$

PROBLEM (II). Find $u\in\mathcal{V}$ such that $u'\in\mathcal{W}$ and

$$\begin{cases}\langle u''(t)+A(t,u'(t))+Bu(t)-f(t),v\rangle_{V^*\times V}\\ \quad+\int_\Gamma j^0(x,t,\gamma u'(t);\gamma v)\,\mathrm{d}\sigma(x)\geqslant 0 \quad \text{for all } v\in V \text{ and a.e. } t\in(0,T)\\ u(0)=u_0,\ \ u'(0)=u_1.\end{cases}$$

The hypotheses on the data are the following:

$\underline{H(A)}$: $A\colon(0,T)\times V\to V^*$ is an operator such that

(i) $t\to\langle A(t,u),v\rangle$ is measurable on $(0,T)$ for all $u,\ v\in V$;
(ii) $\|A(t,v)\|_{V^*}\leqslant a(t)+b\|v\|$ a.e. t, for $v\in V$ with $a\in L^2(0,T)$, $a\geqslant 0$, $b>0$;
(iii) $\langle A(t,v),v\rangle\geqslant\alpha\|v\|^2$ a.e. $t\in(0,T)$, for all $v\in V$ with $\alpha>0$;
(iv) $v\to A(t,v)$ is pseudomonotone for every $t\in(0,T)$.

$\underline{H(B)}$: $B\colon V\to V^*$ is a bounded, linear, monotone and symmetric operator, (i.e. $B\in\mathcal{L}(V,V^*)$, $\langle Bv,v\rangle\geqslant 0$ for all $v\in V$, $\langle Bv,w\rangle=\langle Bw,v\rangle$ for all $v,\ w\in V$).

$\underline{H(j)}$: $j\colon\Gamma\times(0,T)\times\mathbb{R}^N\to\mathbb{R}$ is a function such that

(i) $j(\cdot,\cdot,\xi)$ is measurable for all $\xi\in\mathbb{R}^N$ and $j(\cdot,t,0)\in L^1(\Gamma)$;
(ii) $j(x,t,\cdot)$ is locally Lipschitz for all $x\in\Gamma$, $t\in(0,T)$;
(iii) $|\eta|_{\mathbb{R}^N}\leqslant c(1+|\xi|_{\mathbb{R}^N})$ for all $\eta\in\partial j(x,t,\xi)$, $t\in(0,T)$, $x\in\Gamma$ with $c>0$.

$\underline{(H_0)}$: $f\in\mathcal{V}^*$, $u_0\in V$, $u_1\in H$.

In the hypothesis $H(j)$ the symbol ∂j denotes the Clarke subdifferential of j with respect to the variable ξ.

We begin with an auxiliary result on the properties of the functional $J\colon (0,T)\times L^2(\Gamma;\mathbb{R}^N)\to\mathbb{R}$ given by

$$J(t,v)=\int_\Gamma j(x,t,v(x))\,\mathrm{d}\sigma(x),\quad t\in(0,T),\ \ v\in L^2(\Gamma;\mathbb{R}^N). \tag{1}$$

LEMMA 2. *Assume that* $j\colon \Gamma\times(0,T)\times\mathbb{R}^N\to\mathbb{R}$ *satisfies hypothesis* $H(j)$. *Then the functional* $J(t,\cdot)\colon L^2(\Gamma;\mathbb{R}^N)\to\mathbb{R}$ *given by* (1) *is well defined and locally Lipschitz* (*in fact, Lipschitz on bounded subsets of* $L^2(\Gamma;\mathbb{R}^N)$)*, its generalized gradient satisfies*

$$\zeta\in\partial J(t,v)\implies \|\zeta\|_{L^2(\Gamma;\mathbb{R}^N)}\leqslant c_1\left(1+\|v\|_{L^2(\Gamma;\mathbb{R}^N)}\right), \tag{2}$$

where

$$c_1=\sqrt{2}c\max\{1,\sqrt{\mathrm{meas}_{N-1}(\Gamma)}\} \tag{3}$$

and for its generalized directional derivative we have

$$J^0(t,u;v)\leqslant\int_\Gamma j^0(x,t,u(x);v(x))\,\mathrm{d}\sigma(x)\quad \textit{for } t\in(0,T),\ u,v\in L^2(\Gamma;\mathbb{R}^N). \tag{4}$$

Moreover, if additionally either $j(x,t,\cdot)$ *or* $-j(x,t,\cdot)$ *is regular in the sense of Clarke, then* $J(t,\cdot)$ *or* $-J(t,\cdot)$ *is regular, respectively,*

$$J^0(t,u;v)=\int_\Gamma j^0(x,t,u(x);v(x))\,\mathrm{d}\sigma(x)\quad \text{for } t\in(0,T),\ u,v\in L^2(\Gamma;\mathbb{R}^N) \tag{5}$$

and

$$\partial(J\circ\gamma)(t,v)=\gamma^*\circ\partial J(t,\gamma v)\quad \textit{for } t\in(0,T),\ \ v\in V, \tag{6}$$

where $\gamma^*\colon L^2(\Gamma;\mathbb{R}^N)\to V^*$ *denotes the adjoint operator of* γ *given by*

$$\gamma^* z(v)=\int_\Gamma z(x)\,\gamma v(x)\,\mathrm{d}\sigma(x)\quad \text{for } v\in V \text{ and } z\in L^2(\Gamma;\mathbb{R}^N).$$

Proof. The well posedness of J follows from Theorem 2 of Aubin and Clarke [2]. The estimate in (2) and the inequality (4) are consequences of $H(j)$(iii) and the Fatou lemma, respectively. If either $j(x,t,\cdot)$ or $-j(x,t,\cdot)$ is regular, we have

$$\int_\Gamma j^0(x,t,u(x);v(x))\mathrm{d}\sigma(x) \leqslant J^0(t,u;v) \quad \text{for } t\in(0,T),\ u,v\in L^2(\Gamma;\mathbb{R}^N)$$

which together with (4) entails (5). Finally, since $\gamma\colon V\to L^2(\Gamma;\mathbb{R}^N)$ is linear and continuous, we can apply the chain rule (see Theorem 2.3.10 of Clarke [8]) to calculate the subdifferential of the composition which, in view of the regularity of either $J(x,t,\cdot)$ or $-J(x,t,\cdot)$, implies the equality (6). $\square$

First we consider the problem (I) for which we introduce an operator inclusion which will be solved using a surjectivity result. We associate with (I) the following auxillary inequality problem:

$$\begin{cases} \text{find } u\in\mathcal{V} \text{ with } u'\in\mathcal{W} \text{ such that} \\ \langle u''(t)+A(t,u'(t))+Bu(t)-f(t),v\rangle_{V^*\times V}+J^0(t,\gamma u(t);\gamma v)\geqslant 0 \\ \quad \text{for all } v\in V, \quad \text{a.e. } t\in(0,T) \\ u(0)=u_0,\ u'(0)=u_1, \end{cases} \tag{7}$$

where $J^0(t,u;v)$ denotes the directional derivative of $J(t,\cdot)$ at a point $u\in L^2(\Gamma;\mathbb{R}^N)$ in the direction $v\in L^2(\Gamma;\mathbb{R}^N)$.

In what follows we need the space $Z=H^\delta(\Omega;\mathbb{R}^N)$ with a fixed $\delta\in(\frac{1}{2},1)$. Denoting by $i\colon V\to Z$ the embedding injection and by $\bar\gamma\colon Z\to L^2(\Gamma;\mathbb{R}^N)$ the trace operator, for all $v\in V$ we have $\gamma v=\bar\gamma(iv)$. For simplicity we omit the notation of the embedding i and we write $\gamma v=\bar\gamma v$ for $v\in V$. So we have $V\subset Z\subset H\subset Z^*\subset V^*$ with all embeddings being compact. This also implies that $\mathcal{W}\subset\mathcal{V}\subset\mathcal{Z}\subset\mathcal{H}\subset\mathcal{Z}^*\subset\mathcal{V}^*$, where $\mathcal{Z}=L^2(0,T;Z)$. We consider now the following inclusion:

$$\begin{cases} \text{find } u\in\mathcal{V} \text{ with } u'\in\mathcal{W} \text{ such that} \\ u''(t)+A(t,u'(t))+Bu(t)+\bar\gamma^*(\partial J(t,\bar\gamma u(t)))\ni f(t) \quad \text{a.e. } t\in(0,T) \\ u(0)=u_0,\ u'(0)=u_1. \end{cases} \tag{8}$$

DEFINITION 3. A function $u\in\mathcal{V}$ solves (8) if and only if $u'\in\mathcal{W}$ and there exists $\eta\in\mathcal{Z}^*$ such that

$$\begin{cases} u''(t)+A(t,u'(t))+Bu(t)+\eta(t)=f(t) \quad \text{a.e. } t\in(0,T) \\ \eta(t)\in\bar\gamma^*(\partial J(t,\bar\gamma u(t))) \quad \text{a.e. } t\in(0,T) \\ u(0)=u_0,\ u'(0)=u_1. \end{cases}$$

REMARK 4. We notice that if hypothesis $H(j)$ holds, then every solution to problem (7) is a solution to the problem (I) and every solution to

(8) is also a solution to (7). These facts are easy consequences of inequality (4) of Lemma 2 and the definition of the Clarke subdifferential. If additionally either $j(x,t,\cdot)$ or $-j(x,t,\cdot)$ is regular, then the problems (I), (7) and (8) are equivalent (cf. (5) and (6) of Lemma 2).

4. Existence Results

In this section we deliver existence results for problems (I) and (II). First we consider the problem (I) and in view of Remark 4, we establish the existence result for the problem (8). We start with a priori estimate for solutions of the problem (8).

LEMMA 5. *Suppose that hypotheses $H(A)$, $H(B)$, $H(j)$, (H_0) hold and*

$$\frac{\alpha}{2} > c_1 \beta^2 T \|\bar{\gamma}\|^2, \tag{H_1}$$

where $\beta > 0$ is the embedding constant of V into Z and c_1 is given by (3). *If u is a solution to* (8), *then there exists a constant $C > 0$ such that*

$$\|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \le C\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*})\,. \tag{9}$$

Proof. Let u be a solution to (8). We notice that $u \in C(0,T;V)$ and $u(t) = u_0 + \int_0^t u'(s)\,\mathrm{d}s$ in V with $u' \in \mathcal{W}$. Taking the duality brackets with $u'(t) \in V$ and integrating over $(0,t)$ for any $t \in (0,T)$, we have

$$\begin{aligned}
&\int_0^t \langle u''(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s + \int_0^t \langle A(s,u'(s)), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s \\
&\quad + \int_0^t \langle Bu(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s + \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s \\
&\quad = \int_0^t \langle f(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s
\end{aligned}$$

with $\xi(s) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma} u(t)))$ for a.e. $s \in (0,t)$. From the integration by parts formula (Proposition 23.23(iv), pp. 422–423 of Zeidler [38]), we get

$$\int_0^t \langle u''(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s = \frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u_1|^2.$$

Since B is symmetric and monotone, it follows that

$$\begin{aligned}\int_0^t \langle Bu(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s &= \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\langle Bu(s), u(s)\rangle_{V^*\times V}\,\mathrm{d}s\\ &= \frac{1}{2}\langle Bu(t), u(t)\rangle_{V^*\times V} - \frac{1}{2}\langle Bu_0, u_0\rangle_{V^*\times V}\\ &\geqslant -\frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u_0\|^2.\end{aligned}$$

Moreover, using the Young inequality with some $\varepsilon > 0$, we have

$$\begin{aligned}\int_0^t \langle f(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s &\leqslant \int_0^t \|f(s)\|_{V^*}\|u'(s)\|\,\mathrm{d}s\\ &\leqslant \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon^2}\|f\|^2_{\mathcal{V}^*}.\end{aligned}$$

From the above bounds and using the coercivity of A (see H(A)(iii)), we obtain

$$\begin{aligned}&\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u_1|^2 + \alpha\|u'\|^2_{L^2(0,t;V)} - \frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u_0\|^2\\ &\quad\leqslant \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon^2}\|f\|^2_{\mathcal{V}^*} - \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s \end{aligned}\tag{10}$$

for all $t \in (0,T)$, where $\xi(s) = \bar{\gamma}^* w(s)$ and $w(s) \in \partial J(s, \bar{\gamma}u(s))$ for a.e. $s \in (0,t)$. Using the inequality $\|u(s)\| \leqslant \|u_0\| + \int_0^s \|u'(\tau)\|\,\mathrm{d}\tau$, the estimate (2) and again the Young inequality, we get

$$\begin{aligned}&\left|\int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s\right|\\ &\quad\leqslant \int_0^t |\langle w(s), \bar{\gamma}u'(s)\rangle_{L^2(\Gamma;\mathbb{R}^N)}|\,\mathrm{d}s\\ &\quad\leqslant \int_0^t c_1(1+\beta\|\bar{\gamma}\|\,\|u(s)\|)\beta\|\bar{\gamma}\|\,\|u'(s)\|\,\mathrm{d}s\\ &\quad\leqslant \int_0^t c_1\left[(1+\beta\|\bar{\gamma}\|\,\|u_0\|) + \beta\|\bar{\gamma}\|\int_0^s \|u'(\tau)\|\,\mathrm{d}\tau\right]\|\bar{\gamma}\|\beta\|u'(s)\|\,\mathrm{d}s\\ &\quad\leqslant \int_0^t \left[\frac{\varepsilon^2}{2}\|u'(s)\|^2 + \frac{1}{2\varepsilon^2}(c_1\beta\|\bar{\gamma}\|(1+\beta\|\bar{\gamma}\|\|u_0\|))^2\right]\mathrm{d}s +\\ &\qquad + c_1\beta^2\|\bar{\gamma}\|^2\left(\int_0^t \|u'(s)\|\,\mathrm{d}s\right)^2,\end{aligned}$$

where $\beta > 0$ is such that $\|v\|_Z \leqslant \beta\|v\|_V$ for all $v \in V$ and $\varepsilon > 0$. By virtue of the Jensen inequality applied to the latter term, we have

$$\begin{aligned}
&\left| \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\, \mathrm{d}s \right| \\
&\quad \leqslant \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon^2}(c_1\beta\|\bar{\gamma}\|(1+\beta\|\bar{\gamma}\|\|u_0\|)^2) \\
&\qquad + c_1\beta^2 T\|\bar{\gamma}\|^2\|u'\|^2_{L^2(0,t;V)}.
\end{aligned}$$

Hence and from (10) we get

$$\begin{aligned}
&\frac{1}{2}|u'(t)|^2 + \left(\alpha - \frac{\varepsilon^2}{2}\right)\|u'\|^2_{L^2(0,t;V)} \\
&\quad \leqslant \frac{1}{2}|u_1|^2 + \frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u_0\|^2 + \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} \\
&\qquad + \frac{1}{2\varepsilon^2}[\|f\|^2_{\mathcal{V}^*} + c_1\beta\|\bar{\gamma}\|(1+\beta\|\bar{\gamma}\|\|u_0\|)^2] + c_1\beta^2 T\|\bar{\gamma}\|^2\|u'\|^2_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in (0,T)$. We choose $\varepsilon > 0$ such that $\alpha - \varepsilon^2 = \frac{\alpha}{2}$. Thus for such ε and from the hypothesis (H_1), we obtain

$$\|u'\|_{L^2(0,t;V)} \leqslant c_2(1 + \|u_0\| + |u_1| + \|f\|v^*) \tag{11}$$

with $c_2 > 0$. Hence, we get

$$\|u(t)\| \leqslant \|u_0\| + \int_0^t \|u'(s)\|\mathrm{d}s \leqslant \|u_0\| + \sqrt{T}c_2(1 + \|u_0\| + |u_1| + \|f\|v^*)$$

which implies

$$\|u\|_{C(0,T;V)} \leqslant c_3(1 + \|u_0\| + |u_1| + \|f\|v^*) \quad \text{with } c_3 > 0. \tag{12}$$

To end the proof it is enough to show the bound on $\|u''\|_{\mathcal{V}^*}$. Since u is a solution to (8), from $H(A)$(ii), $H(B)$ and (2), we have

$$\begin{aligned}
\|u''\|_{\mathcal{V}^*} \leqslant \|f\|_{\mathcal{V}^*} &+ \bar{a}_1 + \bar{b}_1\|u'\|_{\mathcal{V}} + \|B\|_{\mathcal{L}(V,V^*)}\|u\|_{\mathcal{V}} \\
&+ \tilde{\beta}c_1\|\bar{\gamma}\|(1+\beta\|\bar{\gamma}\|\|u\|_{\mathcal{V}})
\end{aligned} \tag{13}$$

where $\tilde{\beta}$ is the embedding constant of $\mathcal{Z}^*$ into $\mathcal{V}^*$, $\bar{a}_1 = \sqrt{2}\|a_1\|_{L^2(0,T)}$ and $\bar{b}_1 = \sqrt{2}b_1^2$. Combining (11), (12) and (13), we readily deduce (9). □

THEOREM 6. *If hypotheses* $H(A)$, $H(B)$, $H(j)$, (H_0) *and* (H_1) *hold, then the problem* (8) *has a solution.*

Proof. First we reduce the order of the problem (8). Consider the operator $K\colon \mathcal{V} \to C(0,T;V)$ defined by $Kv(t) = \int_0^t v(\tau)\,d\tau + u_0$ for $v \in \mathcal{V}$. The problem (8) can be formulated as follows

$$\begin{cases} \text{find } z \in \mathcal{W} \text{ such that} \\ z'(t) + A(t, z(t)) + B(Kz(t)) + \bar{\gamma}^*(\partial J(t, \bar{\gamma}(Kz(t)))) \ni f(t) \\ \quad \text{a.e. } t \in (0,T) \\ z(0) = u_1. \end{cases} \tag{14}$$

We observe that z is a solution to (14) if and only if $u = Kz$ satisfies (8). Therefore, in what follows, we will show the existence of solutions to (14). To this end we apply a surjectivity result for L pseudomonotone operators (see Proposition 1). First we consider the case with regular initial condition $u_1 \in V$.

Step 1. Let us assume that $u_1 \in V$. Performing a translation by the initial condition, we define the following operators

$$\begin{cases} \mathcal{A}_1\colon \mathcal{V} \to \mathcal{V}^* \\ (\mathcal{A}_1 v)(t) = A(t, v(t) + u_1) \quad \text{for } v \in \mathcal{V}, \end{cases} \tag{15}$$

$$\begin{cases} \mathcal{B}_1\colon \mathcal{V} \to \mathcal{V}^* \\ (\mathcal{B}_1 v)(t) = B(K(v + u_1)(t)) \quad \text{for } v \in \mathcal{V}, \end{cases} \tag{16}$$

$$\begin{cases} \mathcal{N}_1\colon \mathcal{V} \to 2^{\mathcal{V}^*} \\ \mathcal{N}_1 v = \{ w \in \mathcal{Z}^* :\ w(t) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma}(K(v + u_1)(t)))) \\ \qquad\qquad \text{a.e. } t \in (0,T) \}. \end{cases} \tag{17}$$

Here $v + u_1$ is understood as follows $(v + u_1)(t) = v(t) + u_1$. Let us observe that $\mathcal{A}_1 v = \mathcal{A}(v + u_1)$ and $\mathcal{B}_1 v = \mathcal{B}(K(v + u_1))$, where $\mathcal{A}$ and $\mathcal{B}$ are the Nemytski operators corresponding to A and B, respectively, i.e.

$$(\mathcal{A}v)(t) = A(t, v(t)), \quad (\mathcal{B}v)(t) = B(v(t)) \quad \text{for } v \in \mathcal{V}. \tag{18}$$

In a consequence, from (14) we obtain the following inclusion

$$\begin{cases} z' + \mathcal{A}_1 z + \mathcal{B}_1 z + \mathcal{N}_1 z \ni f \\ z(0) = 0. \end{cases} \tag{19}$$

and note that $z \in \mathcal{W}$ solves (14) if and only if $z - u_1 \in \mathcal{W}$ solves (19). Let us consider the operator $L\colon D(L) \subset \mathcal{V} \to \mathcal{V}^*$ defined by $Lv = v'$ with $D(L) = \{v \in \mathcal{W}:\ v(0) = 0\}$. Recall (see e.g. Zeidler [38], Proposition 32.10, p. 855) that L is a linear, densely defined and maximal monotone operator. Let the operator $\mathcal{T}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ be given by

$$\mathcal{T}z = \mathcal{A}_1 z + \mathcal{B}_1 z + \mathcal{N}_1 z.$$

Now the problem (19) reads as follows:

$$\text{find } z \in D(L) \quad \text{such that } Lz + \mathcal{T}z \ni f.$$

In order to show the existence of a solution to (19) we will prove that the operator $\mathcal{T}$ is bounded, coercive and L pseudomonotone, and apply Proposition 1. We will state the following three lemmas on the properties of the operators $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{N}_1$, respectively. The proofs will be postponed to Section 5.

LEMMA 7.
If $H(A)$ holds and $u_1 \in V$, then the operator $\mathcal{A}_1$ defined by (15) *satisfies:*

(a) $\|\mathcal{A}_1 v\|_{\mathcal{V}^*} \leqslant \hat{a}_1 + \hat{b}_1 \|v\|_{\mathcal{V}}$ *for all* $v \in \mathcal{V}$ *with* $\hat{a}_1 \geqslant 0$ *and* $\hat{b}_1 > 0$;
(b) $\langle\langle \mathcal{A}_1 v, v\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant \frac{\alpha}{2}\|v\|^2_{\mathcal{V}} - \hat{\beta}_2 \|v\|_{\mathcal{V}} - \hat{\beta}_3$ *for all* $v \in \mathcal{V}$ *with* $\hat{\beta}_2 \geqslant 0$ *and* $\hat{\beta}_3 \geqslant 0$;
(c) $\mathcal{A}_1$ *is demicontinuous;*
(d) $\mathcal{A}_1$ *is L pseudomonotone*; *If $H(A)$ holds, then the operator $\mathcal{A}$ defined by* (18) *satisfies*
(e) *For every sequence* $\{v_n\} \subset \mathcal{W}$ *with* $v_n \to v$ *weakly in* $\mathcal{W}$ *and* $\limsup \langle\langle \mathcal{A}v_n, v_n - v\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \leqslant 0$, *it follows that* $\mathcal{A}v_n \to \mathcal{A}v$ *weakly in* $\mathcal{V}^*$ *and* $\langle\langle \mathcal{A}v_n, v_n\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle\langle \mathcal{A}v, v\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}$.

LEMMA 8.
If $H(B)$ holds and $u_1 \in V$, then the operator $\mathcal{B}_1$ defined by (16) *satisfies:*

(a) $\|\mathcal{B}_1 v\|_{\mathcal{V}^*} \leqslant \hat{c}_1(1 + \|v\|_{\mathcal{V}})$ *for all* $v \in \mathcal{V}$ *with* $\hat{c}_1 > 0$;
(b) $\|\mathcal{B}_1 v - \mathcal{B}_1 w\|_{\mathcal{V}^*} \leqslant \hat{c}_2 \|v - w\|_{\mathcal{V}}$ *for all* $v, w \in \mathcal{V}$ *with* $\hat{c}_2 > 0$;
(c) $\langle\langle \mathcal{B}_1 v, v\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant -\hat{c}_3 \|v\|_{\mathcal{V}} - \hat{c}_4$ *for all* $v \in \mathcal{V}$ *with* $\hat{c}_3 \geqslant 0$ *and* $\hat{c}_4 \geqslant 0$;
(d) $\mathcal{B}_1$ *is monotone;*
(e) $\mathcal{B}_1$ *is weakly continuous, i.e., for any sequence* $\{v_n\} \subset \mathcal{V}$ *with* $v_n \to v$ *weakly in* $\mathcal{V}$, *we have* $\mathcal{B}_1 v_n \to \mathcal{B}_1 v$ *weakly in* $\mathcal{V}^*$; *If $H(B)$ holds, then the operator $\mathcal{B}$ defined by* (18) *satisfies*
(f) $\langle\langle \mathcal{B}v - \mathcal{B}w, v' - w'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant 0$ *for all* $v,\ w \in \mathcal{W}$.

LEMMA 9. *If $H(j)$ holds and $u_1 \in V$, then the operator $\mathcal{N}_1$ defined by* (17) *satisfies:*

(a) $\|w\|_{\mathcal{Z}^*} \leqslant \hat{c}_5(1+\|v\|_{\mathcal{V}})$ *for all* $w \in \mathcal{N}_1 v$ *and* $v \in \mathcal{V}$ *with* $\hat{c}_5 > 0$;
(b) *for each* $v \in \mathcal{V}, \mathcal{N}_1 v$ *is a nonempty convex and weakly compact subset of* $\mathcal{Z}^*$;
(c) $\langle\langle w, v\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant -c_1 T \beta^2 \|\bar{\gamma}\|^2 \|v\|^2_{\mathcal{V}} - \hat{c}_6 \|v\|_{\mathcal{V}}$ *for all* $w \in \mathcal{N}_1 v$ *and* $v \in \mathcal{V}$ *with* $\hat{c}_6 > 0$;
(d) *if* $v_n, v \in \mathcal{V}, v_n \to v$ *in* $\mathcal{Z}, w_n, w \in \mathcal{Z}^*, w_n \to w$ *weakly in* $\mathcal{Z}^*$ *and* $w_n \in \mathcal{N}_1 v_n$, *then* $w \in \mathcal{N}_1 v$.

We continue the proof of the theorem.

CLAIM 1. $\mathcal{T}$ is a bounded operator.

The fact that the operator $\mathcal{T}$ maps bounded subsets of $\mathcal{V}$ into bounded subsets of $\mathcal{V}^*$ follows from Lemma 7(a), Lemma 8(a), Lemma 9(a) and the continuity of the embedding $\mathcal{Z}^* \subset \mathcal{V}^*$.

CLAIM 2. $\mathcal{T}$ is coercive.

Let $v \in \mathcal{V}$ and $\eta \in \mathcal{T} v$, i.e., $\eta = \mathcal{A}_1 v + \mathcal{B}_1 v + \xi$ with $\xi \in \mathcal{N}_1 v$. From Lemma 7(b), Lemma 8(c) and Lemma 9(c), we have

$$\begin{aligned}\langle\langle \eta, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle\langle \mathcal{A}_1 v, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle\langle \mathcal{B}_1 v, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle\langle \xi, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &\geqslant \frac{\alpha}{2}\|v\|^2_{\mathcal{V}} - \hat{\beta}_2\|v\|_{\mathcal{V}} - \hat{\beta}_3 - \hat{c}_3\|v\|_{\mathcal{V}} - \hat{c}_4 - c_1 T\beta^2\|\bar{\gamma}\|^2\|v\|^2_{\mathcal{V}} \\ &\quad - \hat{c}_6\|v\|_{\mathcal{V}}.\end{aligned}$$

Due to the hypothesis (H_1), this immediately shows the coercivity of $\mathcal{T}$.

CLAIM 3. $\mathcal{T}$ is L pseudomonotone.

From Lemma 9(b) it follows that for every $v \in \mathcal{V}$, $\mathcal{T} v$ is a nonempty convex and weakly compact subset of $\mathcal{V}^*$. We prove that $\mathcal{T}$ is upper semicontinuous in $\mathcal{V} \times \mathcal{V}^*_{\text{weak}}$ topology. To this end, we show that if a set D is weakly closed in $\mathcal{V}^*$, then the set

$$\mathcal{T}^-(D) = \{v \in \mathcal{V} \colon \mathcal{T} v \cap D \neq \emptyset\} \quad \text{is closed in } \mathcal{V}.$$

Let $\{v_n\} \subset \mathcal{T}^-(D)$ and assume that $v_n \to v$ in $\mathcal{V}$. We can find $\eta_n \in \mathcal{T} v_n \cap D$ for all $n \in \mathbb{N}$ and by definition

$$\eta_n = \mathcal{A}_1 v_n + \mathcal{B}_1 v_n + \xi_n \quad \text{with } \xi_n \in \mathcal{N}_1 v_n. \tag{20}$$

Since $\{v_n\}$ is bounded in $\mathcal{V}$ and $\mathcal{T}$ is a bounded multifunction (cf. Claim 1), then the sequence $\{\eta_n\}$ is bounded in $\mathcal{V}^*$. Hence we may suppose that

$$\eta_n \to \eta \quad \text{weakly in } \mathcal{V}^* \quad \text{with } \eta \in D \tag{21}$$

because D is weakly closed in $\mathcal{V}^*$. Moreover, by Lemma 9(a) we know that $\{\xi_n\}$ is bounded in $\mathcal{Z}^*$ and again we may assume that

$$\xi_n \to \xi \quad \text{weakly in } \mathcal{Z}^* \quad \text{with } \xi \in \mathcal{Z}^*. \tag{22}$$

Hence and from the fact that $v_n \to v$ in $\mathcal{Z}$ (recall that $\mathcal{V} \subset \mathcal{Z}$ continuously), by Lemma 9(d), we obtain $\xi \in \mathcal{N}_1 v$. Next, using the demicontinuity of $\mathcal{A}_1$ (cf. Lemma 7(c)) and the continuity of $\mathcal{B}_1$ (cf. Lemma 8(b)), we have

$$\begin{aligned} &\mathcal{A}_1 v_n \to \mathcal{A}_1 v \quad \text{weakly in } \mathcal{V}^*, \\ &\mathcal{B}_1 v_n \to \mathcal{B}_1 v \quad \text{in } \mathcal{V}^*. \end{aligned}$$

From these convergences, (21) and (22), passing to the limit in (20) we get

$$\eta = \mathcal{A}_1 v + \mathcal{B}_1 v + \xi \quad \text{with } \xi \in \mathcal{N}_1 v$$

which means that $\eta \in \mathcal{T} v \cap D$, so $v \in \mathcal{T}^-(D)$. This proves that $\mathcal{T}^-(D)$ is closed in $\mathcal{V}$, hence $\mathcal{T}$ is upper semicontinuous from $\mathcal{V}$ into $\mathcal{V}^*_{\text{weak}}$.

To conclude the proof that $\mathcal{T}$ is L pseudomonotone, it is enough to show the condition (d) in the definition of pseudomonotonicity (see Preliminaries). Let $\{z_n\} \subset D(L)$, $z_n \to z$ weakly in $\mathcal{W}$, $\eta_n \in \mathcal{T} z_n$, $\eta_n \to \eta$ weakly in $\mathcal{V}^*$ and assume that

$$\limsup \langle\langle \eta_n, z_n - z \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \leqslant 0. \tag{23}$$

So we have $\eta_n = \mathcal{A}_1 z_n + \mathcal{B}_1 z_n + \xi_n$ with $\xi_n \in \mathcal{N}_1 z_n$ for all $n \in \mathbb{N}$.

Since $\mathcal{N}_1$ is a bounded map (cf. Lemma 9(a)) and $\{z_n\}$ is bounded in $\mathcal{V}$, we infer that $\{\xi_n\}$ remains in a bounded subset of $\mathcal{Z}^*$. By passing to a subsequence, if necessary, we may suppose that

$$\xi_n \to \xi \quad \text{weakly in } \mathcal{Z}^*. \tag{24}$$

Since $V \subset Z$ compactly, from Theorem 5.1, Chapter 1 of Lions [15], we have that $\mathcal{W} \subset \mathcal{Z}$ compactly. Thus we may assume that

$$z_n \to z \quad \text{in } \mathcal{Z}. \tag{25}$$

From (24), (25) and Lemma 9(d) we deduce that $\xi \in \mathcal{N}_1 z$. From Lemma 9(a) and (25), we have

$$|\langle\langle \xi_n, z_n - z\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}}| \leqslant \|\xi_n\|_{\mathcal{Z}^*}\|z_n - z\|_{\mathcal{Z}} \leqslant \tilde{c}_5\,(1 + \|z_n\|_{\mathcal{V}})\,\|z_n - z\|_{\mathcal{Z}} \to 0. \tag{26}$$

On the other hand, by the monotonicity of $\mathcal{B}_1$ (cf. Lemma 8(d)) and (25), we obtain

$$\limsup\langle\langle \mathcal{B}_1 z_n, z - z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant \limsup\langle\langle \mathcal{B}_1 z, z - z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0. \tag{27}$$

Combining the condition (23) with (26) and (27), we infer that

$$\begin{aligned}\limsup\langle\langle \mathcal{A}_1 z_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant{} & \limsup\langle\langle \eta_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & + \limsup\langle\langle \mathcal{B}_1 z_n, z - z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & + \limsup\langle\langle \xi_n, z - z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant 0.\end{aligned}$$

From the L pseudomonotonicity of $\mathcal{A}_1$ (cf. Lemma 7(d)), we have

$$\mathcal{A}_1 z_n \to \mathcal{A}_1 z \quad \text{weakly in } \mathcal{V}^* \tag{28}$$

and

$$\begin{aligned}&\langle\langle \mathcal{A}_1 z_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle\langle \mathcal{A}_1 z, z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \text{or equivalently} \\ &\quad \langle\langle \mathcal{A}_1 z_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0.\end{aligned} \tag{29}$$

Hence and from (28), the weak continuity of $\mathcal{B}_1$ (cf. Lemma 8(e)) and (24), we conclude that

$$\eta_n = \mathcal{A}_1 z_n + \mathcal{B}_1 z_n + \xi_n \to \mathcal{A}_1 z + \mathcal{B}_1 z + \xi = \eta \quad \text{weakly in } \mathcal{V}^*.$$

This together with $\xi \in \mathcal{N}_1 z$ implies $\eta \in \mathcal{T} z$.

Moreover, we also have

$$\langle\langle \mathcal{B}_1 z_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle\langle \mathcal{B}_1 z, z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \tag{30}$$

Indeed, from (23), (26) and (29), we get

$$\begin{aligned}\limsup\langle\langle \mathcal{B}_1 z_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant{} & \limsup\langle\langle \eta_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & - \lim\langle\langle \mathcal{A}_1 z_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & - \lim\langle\langle \xi_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant 0\end{aligned}$$

which together with (27) implies $\lim\langle\langle \mathcal{B}_1 z_n, z_n - z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0$, so also (30). Passing to the limit in the equation

$$\langle\langle \eta_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle\langle \mathcal{A}_1 z_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle\langle \mathcal{B}_1 z_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle\langle \xi_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}$$

from (29), (30) and (26), we obtain $\lim\langle\langle \eta_n, z_n\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle\langle \eta, z\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}$ with $\eta \in \mathcal{T}z$. This proves the L pseudomonotonicity of $\mathcal{T}$.

Since $\mathcal{V}$ is a strictly convex Banach space (this follows from the fact that in every reflexive Banach space there exists an equivalent norm such that this space is strictly convex, see Zeidler [38], p. 256), from Claims 1, 2, 3, by Proposition 1, we deduce that the problem (19) has a solution $z \in D(L)$, so $z + u_1$ solves (14), and $u = K(z + u_1)$ is a solution of (8) in the case when $u_1 \in V$.

Step 2. Let us assume that $u_1 \in H$. By the density of V in H, we can find a sequence $\{u_{1n}\} \subset V$ such that $u_{1n} \to u_1$ in H, as $n \to \infty$. Consider a solution u_n of the problem (8), when u_1 is replaced by u_{1n}, i.e. a solution of the following problem

$$\begin{cases} \text{find} \quad u_n \in \mathcal{V} \text{ such that } u_n' \in \mathcal{W} \text{ and} \\ u_n''(t) + A(t, u_n'(t)) + Bu_n(t) + \bar{\gamma}^*(\partial J(t, \bar{\gamma} u_n(t))) \ni f(t) \quad \text{a.e. } t \in (0, T) \\ u_n(0) = u_0, \ \ u_n'(0) = u_{1n}. \end{cases}$$

The existence of u_n, for $n \in \mathbb{N}$, follows from the first part of the proof. We have

$$u_n''(t) + A(t, u_n'(t)) + Bu_n(t) + \xi_n(t) = f(t) \quad \text{for a.e. } t \in (0, T)$$

or equivalently

$$u_n'' + \mathcal{A}u_n' + \mathcal{B}u_n + \xi_n = f \quad \text{in } \mathcal{V}^* \tag{31}$$

with $\xi_n \in \mathcal{N}u_n$ and the initial conditions $u_n(0) = u_0$, $u_n'(0) = u_{1n}$. Recall that $\mathcal{A}$ and $\mathcal{B}$ are the Nemitsky operators corresponding to A and B, respectively, (cf. (18)) and $\mathcal{N}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is given by

$$\mathcal{N}v = \{w \in \mathcal{Z}^* \colon w(t) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma} v(t))) \text{ a.e. } t \in (0, T)\} \quad \text{for } v \in \mathcal{V}.$$

From Lemma 5, we have

$$\|u_n\|_{C(0,T;V)} + \|u_n'\|_{\mathcal{W}} \leqslant C(1 + \|u_0\| + |u_{1n}| + \|f\|_{\mathcal{V}^*}).$$

Since $\{u_{1n}\}$ is bounded in H, we get that $\{u_n\}$, $\{u_n'\}$ are bounded respectively in $\mathcal{V}$ and $\mathcal{W}$ uniformly with respect to n. Hence, by passing to a subsequence if necessary, we assume

$$\begin{aligned} &u_n \to u \quad \text{weakly in } \mathcal{V}, \\ &u_n' \to u' \quad \text{weakly in } \mathcal{V}, \\ &u_n'' \to u'' \quad \text{weakly in } \mathcal{V}^*. \end{aligned}$$

We will show that u is a solution to the problem (8).

Since $u_n \to u$, $u_n' \to u'$ both weakly in $\mathcal{W}$ and $\mathcal{W} \subset C(0,T;H)$ continuously we get $u_n(t) \to u(t)$ and $u_n'(t) \to u'(t)$ both weakly in H for all $t \in [0,T]$. Hence $u_0 = u_n(0) \to u(0)$ weakly in H, which gives $u(0) = u_0$. Also from the convergences $u_{1n} \to u_1$ in H and $u_{1n} = u_n'(0) \to u'(0)$ weakly in H, we get $u'(0) = u_1$. Next, from $\xi_n \in \mathcal{N} u_n$ we have $\xi_n(t) = \bar{\gamma}^* z_n(t)$ and

$$z_n(t) \in \partial J(t, \bar{\gamma} u_n(t)) \quad \text{a.e. } t \in (0,T). \tag{32}$$

Similarily as in Lemma 9(a), using (2), we get that $\{z_n\}$ remains in a bounded subset of $L^2(0,T;L^2(\Gamma;\mathbb{R}^N))$, and so for a subsequence we may assume

$$z_n \to z \quad \text{weakly in } L^2(0,T;L^2(\Gamma;\mathbb{R}^N)). \tag{33}$$

Moreover, we obtain

$$\xi_n \to \xi \quad \text{weakly in } \mathcal{Z}^*. \tag{34}$$

Using the last two convergences from $\xi_n(t) = \bar{\gamma}^* z_n(t)$ for a.e. $t \in (0,T)$, passing to the limit, we get $\xi(t) = \bar{\gamma}^* z(t)$ a.e. $t \in (0,T)$. Since $\mathcal{W} \subset \mathcal{Z}$ compactly and $u_n \to u$ weakly in $\mathcal{W}$, we obtain $u_n \to u$ in $\mathcal{Z}$, and subsequently $u_n(t) \to u(t)$ in Z for a.e. $t \in (0,T)$ and

$$\bar{\gamma}(u_n(t)) \to \bar{\gamma}(u(t)) \quad \text{in } L^2(\Gamma;\mathbb{R}^N) \quad \text{for a.e. } t \in (0,T). \tag{35}$$

Exploiting (33) and (35) and passing to the limit in (32), we deduce (cf. again the Convergence Theorem of Aubin and Cellina [1]) that $z(t) \in \partial J(t, \bar{\gamma} u(t))$ for a.e. $t \in (0,T)$, which clearly implies $\xi \in \mathcal{N} u$.

The proof now ends with the passing to the limit in (31). For this purpose, by using Lemma 7(e) we will show

$$\mathcal{A} u_n' \to \mathcal{A} u' \quad \text{weakly in } \mathcal{V}^*. \tag{36}$$

Since $\lim \langle\langle f, u_n' - u' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} = 0$ and $\lim \langle\langle \xi_n, u_n' - u' \rangle\rangle_{\mathcal{Z}^* \times \mathcal{Z}} = 0$ (recall $\xi_n \to \xi$ weakly in $\mathcal{Z}^*$ and $u_n' \to u'$ in $\mathcal{Z}$), from (31) we have

$$\begin{aligned} &\lim\ \sup \langle\langle \mathcal{A} u_n', u_n' - u' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad \leqslant \lim\sup \langle\langle u_n'', u' - u_n' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} + \lim\sup \langle\langle \mathcal{B} u_n, u' - u_n' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}. \end{aligned} \tag{37}$$

Using the equality (cf. Proposition 23.23(iv) of Zeidler [38], p. 422)

$$\begin{aligned} \langle\langle u_n'' - u'', u_n' - u' \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} &= \frac{1}{2} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} |u_n'(t) - u'(t)|^2 \, \mathrm{d}t \\ &= \frac{1}{2} |u_n'(T) - u'(T)|^2 - \frac{1}{2} |u_n'(0) - u'(0)|^2, \end{aligned}$$

we obtain

$$\limsup \langle\langle u_n'' - u'', u' - u_n'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \leqslant 0. \tag{38}$$

On the other hand, by property (f) of Lemma 8, we have

$$\begin{aligned} \limsup \langle\langle \mathcal{B}u_n, u' - u_n'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} &= \limsup\, (-\langle\langle \mathcal{B}u - \mathcal{B}u_n, u' - u_n'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad + \langle\langle \mathcal{B}u, u' - u_n'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}) \leqslant \limsup \langle\langle \mathcal{B}u, u' - u_n'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} = 0. \end{aligned} \tag{39}$$

Consequently, by exploiting (38) and (39) in (37), we get $\limsup \langle\langle \mathcal{A}u_n', u_n' - u'\rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \leqslant 0$. Since $u_n' \to u'$ weakly in $\mathcal{W}$, we apply Lemma 7(e) and obtain (36).

By using (34), (36) and the weak continuity of $\mathcal{B}$ (cf. Lemma 8(e)), we pass to the limit in (31) and obtain $u'' + \mathcal{A}u' + \mathcal{B}u + \xi = f$ in $\mathcal{V}^*$. This together with the conditions $\xi \in \mathcal{N}u$, $u(0) = u_0$ and $u'(0) = u_1$ implies that u is a solution to the problem (8). The proof of the theorem is complete. $\square$

In the remaining part of this section, we deal with the problem (II). The approach to prove existence of solutions to (II) is analogous to that we used for problem (I). We formulate the following inclusion:

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ with } u' \in \mathcal{W} \text{ such that} \\ u''(t) + A(t, u'(t)) + Bu(t) + \bar{\gamma}^* \left(\partial J(t, \bar{\gamma} u'(t))\right) \ni f(t) \\ \qquad \text{a.e. } t \in (0, T) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{40}$$

The definition of the solution to (40) can be stated analogously to Definition 3. We also remark that every solution to problem (40) is a solution to (II). We begin with a priori bounds for solutions of (40).

LEMMA 10. *Assume the hypotheses $H(A)$, $H(B)$, $H(j)$, (H_0) hold and*

$$\frac{\alpha}{2} > c_1 \beta^2 \|\bar{\gamma}\|^2, \tag{$\widetilde{H}_1$}$$

where $\beta > 0$ is the embedding constant of V into Z. If u is a solution to (40)*, then*

$$\|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \leq C\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*})\,.$$

with a positive constant C.

Proof. Similarily as in the proof of Lemma 5, we obtain

$$\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u_1|^2 + \alpha\|u'\|^2_{L^2(0,t;V)} - \frac{1}{2}\|B\|_{\mathcal{L}(V,V^*)}\|u_0\|^2$$
$$\leqslant \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon^2}\|f\|^2_{\mathcal{V}^*} - \int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s$$

for all $t\in(0,T)$ and $\varepsilon>0$, where $\xi(s)=\bar{\gamma}^* w(s)$ and $w(s)\in\partial J(s,\bar{\gamma}u'(s))$ for a.e. $s\in(0,t)$. From (2) and the Young inequality, we have

$$\left|\int_0^t \langle \xi(s), u'(s)\rangle_{V^*\times V}\,\mathrm{d}s\right| \le \int_0^t |\langle w(s), \bar{\gamma}u'(s)\rangle_{L^2(\Gamma;\mathbb{R}^N)}|\mathrm{d}s$$
$$\leqslant \int_0^t c_1(1+\beta\|\bar{\gamma}\|\|u'(s)\|)\beta\|\bar{\gamma}\|\|u'(s)\|\mathrm{d}s$$
$$\le \int_0^t (c_1\beta\|\bar{\gamma}\|\|u'(s)\| + c_1\beta^2\|\bar{\gamma}\|^2\|u'(s)\|^2)\,\mathrm{d}s$$
$$\leqslant \frac{\varepsilon^2}{2}\|u'\|^2_{L^2(0,t;V)} + \frac{1}{2\varepsilon^2}Tc_1^2\beta^2\|\bar{\gamma}\|^2 + c_1\beta^2\|\bar{\gamma}\|^2\|u'\|^2_{L^2(0,t;V)}.$$

Choosing $\varepsilon>0$ such that $\alpha-\varepsilon^2=\frac{\alpha}{2}$ and then making use of the hypothesis $(\widetilde{H}_1)$, we obtain

$$\|u'\|_{L^2(0,t;V)} \leqslant c_2\,(1+\|u_0\|+|u_1|+\|f\|_{\mathcal{V}^*})$$

with $c_2>0$. The remaining conclusions of the proof can be done analogously as in Lemma 5. This completes the proof of the lemma. □

THEOREM 11. *Under the assumptions $H(A)$, $H(B)$, $H(j)$, (H_0) and $(\widetilde{H}_1)$, the problem* (40) *has a solution.*

Proof. We proceed as in the proof of Theorem 6. Using the operator $K:\mathcal{V}\to C(0,T;V)$ defined by $Kv(t)=\int_0^t v(\tau)\,d\tau+u_0$ for $v\in\mathcal{V}$, we formulate the problem (40) as follows:

$$\begin{cases} \text{Find } z\in\mathcal{W} \text{ such that} \\ z'(t)+A(t,z(t))+B(Kz(t))+\bar{\gamma}^*(\partial J(t,\bar{\gamma}z(t)))\ni f(t) \\ \quad \text{for a.e. } t\in(0,T) \\ z(0)=u_1. \end{cases} \tag{41}$$

We notice that z is a solution to (41) if and only if $u=Kz$ satisfies (40). To prove the existence to (40), we apply Proposition 1 to the problem (41).

Step 1. Let us suppose that $u_1 \in V$. We define the operators $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{N}_2$ respectively by (15), (16) and

$$\begin{cases} \mathcal{N}_2 : \mathcal{V} \to 2^{\mathcal{V}^*} \\ \mathcal{N}_2 v = \{w \in \mathcal{Z}^* : w(t) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma}(v + u_1)(t))) \\ \qquad\qquad \text{a.e. } t \in (0, T)\}. \end{cases} \tag{42}$$

The problem (41) we rewrite as follows:

$$\begin{cases} z' + \mathcal{A}_1 z + \mathcal{B}_1 z + \mathcal{N}_2 z \ni f \\ z(0) = 0. \end{cases} \tag{43}$$

and remark that $z \in \mathcal{W}$ solves (41) if and only if $z - u_1 \in \mathcal{W}$ solves (43). Next we formulate the problem (43) as an operator inclusion:

$$\text{find } z \in D(L) \quad \text{such that } Lz + \widetilde{\mathcal{T}} z \ni f,$$

where $\widetilde{\mathcal{T}} \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is given by $\widetilde{\mathcal{T}} z = \mathcal{A}_1 z + \mathcal{B}_1 z + \mathcal{N}_2 z$. The following result shows that $\mathcal{N}_2$ has properties analogous to $\mathcal{N}_1$ (cf. Lemma 9).

LEMMA 12. *If $H(j)$ holds and $u_1 \in V$, then the operator $\mathcal{N}_2$ given by* (42) *satisfies:*

(a) $\|w\|_{\mathcal{Z}^*} \leqslant \bar{c}(1 + \|v\|_{\mathcal{V}})$ *for all $w \in \mathcal{N}_2 v$ and $v \in \mathcal{V}$ with* $\bar{c} > 0$;
(b) *for every $v \in \mathcal{V}$, $\mathcal{N}_2 v$ is a nonempty convex and weakly compact subset of* $\mathcal{Z}^*$;
(c) $\langle\langle w, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant -c_1 \beta^2 \|\bar{\gamma}\|^2 \|v\|_{\mathcal{V}}^2 - \hat{c}\, \|v\|_{\mathcal{V}}$ *for all $w \in \mathcal{N}_2 v$ and $v \in \mathcal{V}$ with* $\hat{c} > 0$;
(d) *for every v_n, $v \in \mathcal{V}$ with $v_n \to v$ in $\mathcal{Z}$ and every w_n, $w \in \mathcal{Z}^*$ with $w_n \to w$ weakly in $\mathcal{Z}^*$, if $w_n \in \mathcal{N}_2 v_n$, then $w \in \mathcal{N}_2 v$.*

Using Lemmas 7, 8 and 12, it can be proved (analogously as in Claims 1, 2 and 3 of Theorem 6) that $\widetilde{\mathcal{T}}$ is a bounded, coercive and L pseudomonotone operator. We comment only on the coercivity of $\widetilde{\mathcal{T}}$. Namely, if $v \in \mathcal{V}$ and $\eta \in \widetilde{\mathcal{T}} v$, then $\eta = \mathcal{A}_1 v + \mathcal{B}_1 v + \xi$ with $\xi \in \mathcal{N}_2 v$. By Lemma 7(b), Lemma 8(c) and Lemma 12(c), we obtain

$$\langle\langle \eta, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant \frac{\alpha}{2} \|v\|_{\mathcal{V}}^2 - \hat{\beta}_2 \|v\|_{\mathcal{V}} - \hat{\beta}_3 - \hat{c}_3 \|v\|_{\mathcal{V}} - \hat{c}_4 - c_1 \beta^2 \|\bar{\gamma}\|^2 \|v\|_{\mathcal{V}}^2 - \hat{c} \|v\|_{\mathcal{V}}.$$

Hence and from the hypothesis $(\widetilde{H_1})$ we deduce coercivity of $\widetilde{\mathcal{T}}$. By Proposition 1 we now infer that the problem (43) admits a solution $z \in D(L)$ and subsequently that $u = K(z + u_1)$ solves the problem (40).

Step 2. We assume that $u_1 \in H$. In this case the existence proof uses the a priori estimate of Lemma 10, Lemma 12 and it is the repetition of Step 2 of Theorem 6 with some minor modifications. □

5. Proofs of Lemmas

In this section we give the proofs of properties of the operators $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{N}_1$ and $\mathcal{N}_2$ which we used in the previous section.

Proof of Lemma 7. The property (a) follows easily from $H(A)(ii)$ and (i). For part (b) we have

$$\begin{aligned}\langle\langle \mathcal{A}_1 v, v\rangle\rangle &= \int_0^T (\langle A(t, v(t)+u_1), v(t)+u_1\rangle - \langle A(t, v(t)+u_1), u_1\rangle)\,\mathrm{d}s\\ &\geqslant \alpha\int_0^T \frac{1}{2}\|v(t)\|^2\mathrm{d}t - \|u_1\|^2 - \|a\|_{L^1(0,T)} - b_1\|u_1\|\int_0^T \|v(t)+u_1\|\,\mathrm{d}t\\ &\geqslant \frac{\alpha}{2}\|v\|_{\mathcal{V}}^2 - \hat{\beta}_2\|v\|_{\mathcal{V}} - \hat{\beta}_3.\end{aligned}$$

Here we have used $H(A)(iii)$ and (ii), and the inequality $|a+b|^2 \geqslant \frac{1}{2}|a|^2 - |b|$, a, $b \in \mathbb{R}$.

Proof of (c). By $H(A)(ii)$ the operator $v \to A(t, v)$ is bounded. From Proposition 27.7, p. 588 of Zeidler [38] we know that a pseudomonotone and locally bounded operator is demicontinuous. Now exploiting the demicontinuity of $v \to A(t, v)$ and $H(A)(i)$ and (ii) by Lemma 1 of Berkovitz and Mustonen [3], we obtain that $\mathcal{A}_1$ is demicontinuous.

The property (d) was proved by Berkovitz and Mustonen [3] in Theorem 2(b).

Proof of (e). The condition (e) is close to a condition of pseudomonotonicity of $\mathcal{A}$ except the fact that we do not require the sequence $\{v_n\}$ to be in $D(L)$ but only in $\mathcal{W}$. Since the condition $v_n(0)=0$ does not play any role in the proof of pseudomonotonicity of $\mathcal{A}$, the proof of (e) is a repetition of the one of Theorem 2(b) of Berkovitz and Mustonen [3] (see also Lemma 1.9 of Ochal [29]). □

Proof of Lemma 8. In the proof we use the following properties of the (nonlinear) operator $K : \mathcal{V} \to C(0, T; V)$:

$$\|Kv\|_{C(0,T;V)} \leqslant \sqrt{T}\|v\|_{\mathcal{V}} + \|u_0\| \quad \text{for all } v \in \mathcal{V}, \tag{K_1}$$

$$\|Kv - Kw\|_{C(0,T;V)} \leqslant \sqrt{T}\|v - w\|_{\mathcal{W}} \quad \text{for all } v, w \in \mathcal{V} \tag{K_2}$$

We start with (a). Let $v \in \mathcal{V}$. Using (K_1), we have

$$\begin{aligned}\|\mathcal{B}_1 v\|_{\mathcal{V}^*}^2 &= \int_0^T \|B(K(v+u_1)(t))\|_{V^*}^2\,\mathrm{d}t\\ &\leqslant \|B\|_{\mathcal{L}(V,V^*)}^2 \int_0^T \|K(v+u_1)(t)\|^2\,\mathrm{d}t\\ &\leqslant T\|B\|_{\mathcal{L}(V,V^*)}^2\big(\sqrt{T}\|v+u_1\|_{\mathcal{V}}+\|u_0\|\big)^2.\end{aligned}$$

Hence $\|\mathcal{B}_1 v\|_{\mathcal{V}^*} \leqslant \sqrt{T}\|B\|_{\mathcal{L}(V,V^*)}(\sqrt{T}\|v+u_1\|_{\mathcal{V}}+\|u_0\|)$ and the condition (a) follows. In order to obtain (b), we use (K_2) and for $v,\ w\in\mathcal{V}$ we have

$$\begin{aligned}\|\mathcal{B}_1 v-\mathcal{B}_1 w\|_{\mathcal{V}^*}^2 &= \int_0^T \|B(K(v+u_1)(t))-B(K(w+u_1)(t))\|_{V^*}^2\,\mathrm{d}t\\ &\le \|B\|_{\mathcal{L}(V,V^*)}^2\int_0^T \|K(v+u_1)(t)-K(w+u_1)(t)\|^2\,\mathrm{d}t\\ &\le T^2\|B\|_{\mathcal{L}(V,V^*)}^2\|v-w\|_{\mathcal{V}}^2.\end{aligned}$$

This implies condition (b).

Next, since B is symmetric and positive, and K is bounded (cf. (K_1)), we get

$$\begin{aligned}\langle\langle\mathcal{B}_1 v, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle B(K(v+u_1)(t)),\,(K(v+u_1))'(t)-u_1\rangle_{V^*\times V}\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle B(K(v+u_1)(t)),\,K(v+u_1)(t)\rangle_{V^*\times V}\,\mathrm{d}t\\ &\quad -\int_0^T \langle B(K(v+u_1)(t)),\,u_1\rangle_{V^*\times V}\,\mathrm{d}t\\ &\geqslant -T\|B\|_{\mathcal{L}(V,V^*)}\|u_1\|\,\|K(v+u_1)\|_{C(0,T;V)}\\ &\geqslant -T\|B\|_{\mathcal{L}(V,V^*)}\|u_1\|\big(\sqrt{T}\|v+u_1\|_{\mathcal{V}}+\|u_0\|\big)\end{aligned}$$

for all $v\in\mathcal{V}$, which proves the property (c).

Using the monotonicity and symmetry of B and the product rule we obtain for all $v,\ w\in\mathcal{V}$

$$\begin{aligned}\langle\langle\mathcal{B}_1 v-\mathcal{B}_1 w, v-w\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle B(K(v+u_1)(t))-B(K(w+u_1)(t)),\\ &\qquad (K(v+u_1))'(t)-(K(w+u_1))'(t)\rangle_{V^*\times V}\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle B(K(v+u_1)(t))-B(K(w+u_1)(t)),\\ &\qquad K(v+u_1)(t)-K(w+u_1)(t)\rangle_{V^*\times V}\,\mathrm{d}t\\ &= \frac{1}{2}\langle B(K(v+u_1)(T))-B(K(w+u_1)(T)),\\ &\qquad K(v+u_1)(T)-K(w+u_1)(T)\rangle_{V^*\times V}\geqslant 0,\end{aligned}$$

which shows the monotonicity of $\mathcal{B}_1$.

Now we show that $\mathcal{B}_1$ is weakly continuous. Let $\{v_n\}$ be a sequence in $\mathcal{V}$ such that $v_n \to v$ weakly in $\mathcal{V}$. Since the operator $G\colon \mathcal{V} \to V$ given by $(Gw)(t) = \int_0^t w(s)\,\mathrm{d}s$ for all $w \in \mathcal{V}$ and $t \in (0,T)$ is linear and continuous, we have $\int_0^t v_n(s)\,\mathrm{d}s \to \int_0^t v(s)\,\mathrm{d}s$ weakly in V. Hence for $t \in (0,T)$ we have $K(v_n+u_1)(t) \to K(v+u_1)(t)$ weakly in V and $B(K(v_n+u_1)(t)) \to B(K(v+u_1)(t))$ weakly in V^*. In view of the condition (a), we can apply the dominated convergence theorem and we get

$$\begin{aligned}\langle\langle \mathcal{B}_1 v_n, \varphi\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle B(K(v_n+u_1)(t)), \varphi(t)\rangle_{V^*\times V}\,\mathrm{d}t\\ &\to \int_0^T \langle B(K(v+u_1)(t)), \varphi(t)\rangle_{V^*\times V}\,\mathrm{d}t = \langle\langle \mathcal{B}_1 v, \varphi\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}\end{aligned}$$

for all $\varphi \in \mathcal{V}$. Hence $\mathcal{B}_1 v_n \to \mathcal{B}_1 v$ weakly in $\mathcal{V}^*$.

To prove property (f), we observe that by the positivity and symmetry of B, for v, $w \in \mathcal{W}$ we have

$$\begin{aligned}\langle\langle \mathcal{B}v - \mathcal{B}w, v' - w'\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle Bv(t) - Bw(t), v(t) - w(t)\rangle_{V^*\times V}\,\mathrm{d}t\\ &= \frac{1}{2}\langle Bv(T) - Bw(T), v(T) - w(T)\rangle_{V^*\times V} \geqslant 0.\end{aligned}$$

This completes the proof of Lemma 8. □

Proof of Lemma 9. Proof of (a). Let $v \in \mathcal{V}$ and $w \in \mathcal{N}_1 v$. Hence $w(t) = \bar{\gamma}^* z(t)$ and $z(t) \in \partial J(t, \bar{\gamma}(K(v+u_1)(t)))$ for a.e. $t \in (0,T)$. Using (2) and (K_1) we have

$$\begin{aligned}\|z(t)\|_{L^2(\Gamma;\mathbb{R}^N)} &\leqslant c_1\left(1 + \|\bar{\gamma}(K(v+u_1)(t))\|_{L^2(\Gamma)}\right)\\ &\leqslant c_1\,(1 + \|\bar{\gamma}\|\|K(v+u_1)(t)\|_Z) \leqslant c_1\big(1 + \beta\|\bar{\gamma}\|\|K(v+u_1)(t)\|\big)\\ &\leqslant c_1\big(1 + \beta\|\bar{\gamma}\|\big(\sqrt{T}\|v+u_1\|_{\mathcal{V}} + \|u_0\|\big)\big).\end{aligned}$$

Hence

$$\begin{aligned}\|w\|_{\mathcal{Z}^*} &= \left(\int_0^T \|\bar{\gamma}^* z(t)\|^2_{Z^*}\,\mathrm{d}t\right)^{1/2} \leqslant \|\bar{\gamma}^*\|\left(\int_0^T \|z(t)\|^2_{L^2(\Gamma;\mathbb{R}^N)}\,\mathrm{d}t\right)^{1/2}\\ &\leqslant c_1\sqrt{T}\|\bar{\gamma}^*\|\big(1 + \beta\|\bar{\gamma}\|\big(\sqrt{T}\|v+u_1\|_{\mathcal{V}} + \|u_0\|\big)\big) \qquad (44)\end{aligned}$$

which proves the property (a).

We now demonstrate the property (b). It is well known (see Proposition 2.1.2 of Clarke [8]) that the values of $\partial J(t,\cdot)$ are nonempty, weakly compact and convex subsets of $L^2(\Gamma;\mathbb{R}^N)$. Hence for every $v \in \mathcal{V}$ the set $\mathcal{N}_1 v$ is nonempty and convex in $\mathcal{Z}^*$. To prove that $\mathcal{N}_1 v$ is weakly compact in $\mathcal{Z}^*$, we will show that it is closed in $\mathcal{Z}^*$. In fact, let $v \in \mathcal{V}$, $\{w_n\} \subset \mathcal{N}_1 v$, $w_n \to w$ in $\mathcal{Z}^*$. Then passing to a subsequence if necessary, we have $w_n(t) \to w(t)$ in Z^* for a.e. $t \in (0, T)$. Since

$$w_n(t) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma}(K(v+u_1)(t)))) \quad \text{for a.e. } t \in (0, T)$$

and the latter is a closed subset of Z^*, we get

$$w(t) \in \bar{\gamma}^*(\partial J(t, \bar{\gamma}(K(v+u_1)(t)))) \quad \text{for a.e. } t \in (0, T).$$

Hence $w \in \mathcal{N}_1 v$. Consequently, the set $\mathcal{N}_1 v$ is closed in $\mathcal{Z}^*$ and convex, so it is also weakly closed in $\mathcal{Z}^*$. Since (cf. the property (a)) $\mathcal{N}_1 v$ is a bounded set in a reflexive Banach space $\mathcal{Z}^*$, we obtain that $\mathcal{N}_1 v$ is weakly compact in $\mathcal{Z}^*$.

Next, we will show (c). Let $v \in \mathcal{V}$ and $w \in \mathcal{N}_1 v$. So $w(t) = \bar{\gamma}^* z(t)$ and $z(t) \in \partial J(t, \bar{\gamma}(K(v+u_1)(t)))$ for a.e. $t \in (0, T)$. Exploiting the inequality (44), we get

$$\begin{aligned}
|\langle\langle w, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}}| = |\langle\langle w, v\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}}| &\leqslant \beta\|w\|_{\mathcal{Z}^*}\|v\|_{\mathcal{V}} \\
&\leqslant c_1\sqrt{T}\beta\|v\|_{\mathcal{V}}\|\bar{\gamma}^*\|(1+\beta\|\bar{\gamma}\|\sqrt{T}\|v\|_{\mathcal{V}}+\beta\|\bar{\gamma}\|\sqrt{T}\|u_1\| \\
&\quad +\beta\|\bar{\gamma}\|\|u_0\|) \\
&\leqslant c_1 T\beta^2\|\bar{\gamma}\|^2\|v\|_{\mathcal{V}}^2 + \tilde{c}\,\|v\|_{\mathcal{V}}
\end{aligned}$$

with $\tilde{c} > 0$. Hence

$$\langle\langle w, v\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} \geqslant -c_1 T\beta^2\|\bar{\gamma}\|^2\|v\|_{\mathcal{V}}^2 - \tilde{c}\,\|v\|_{\mathcal{V}}.$$

In order to prove (d), let $w_n \in \mathcal{N}_1 v_n$ with v_n, $v \in \mathcal{V}$, $v_n \to v$ in $\mathcal{Z}$ and w_n, $w \in \mathcal{Z}^*$, $w_n \to w$ weakly in $\mathcal{Z}^*$. First, by the definition of K and the Jensen inequality, we have

$$\begin{aligned}
&\|K(v_n+u_1) - K(v+u_1)\|_{\mathcal{Z}}^2 \\
&= \int_0^T \left\| \int_0^t v_n(s)\,\mathrm{d}s + u_1 t + u_0 - \int_0^t v(s)\,\mathrm{d}s - u_1 t - u_0 \right\|_Z^2 \mathrm{d}t \\
&= \int_0^T \left\| \int_0^t (v_n(s) - v(s))\,\mathrm{d}s \right\|_Z^2 \mathrm{d}t \leqslant T^2 \int_0^T \|v_n(s) - v(s)\|_Z^2\,\mathrm{d}s = T^2\|v_n - v\|_{\mathcal{Z}}^2.
\end{aligned}$$

Hence $K(v_n+u_1) \to K(v+u_1)$ in $\mathcal{Z}$ and passing to a subsequence if necessary, we have $K(v_n+u_1)(t) \to K(v+u_1)(t)$ in Z for a.e. $t \in (0, T)$. This entails that

$$\bar{\gamma}(K(v_n+u_1)(t)) \to \bar{\gamma}(K(v+u_1)(t)) \quad \text{in } L^2(\Gamma; \mathbb{R}^N) \quad \text{for a.e. } t \in (0, T) \tag{45}$$

Next, we have

$$w_n(t) = \bar{\gamma}^* z_n(t) \tag{46}$$

with

$$z_n(t) \in \partial J(t, \bar{\gamma}(K(v_n + u_1)(t)))) \quad \text{for a.e. } t \in (0, T). \tag{47}$$

From (45), (47) and (2) we deduce that $\{z_n\}$ is bounded in $L^2(0, T; L^2(\Gamma; \mathbb{R}^N))$ and so we may suppose that

$$z_n \to z \quad \text{weakly in } L^2(0, T; L^2(\Gamma; \mathbb{R}^N)) \tag{48}$$

for some $z \in L^2(0, T; L^2(\Gamma; \mathbb{R}^N))$. Using (48) and the convergence $w_n \to w$ weakly in $\mathcal{Z}^*$, from (46) we obtain $w(t) = \bar{\gamma}^* z(t)$ for a.e. $t \in (0, T)$. Moreover, taking into account (45), (48) and the fact that $\partial J(t, \cdot)$ is upper semicontinuous with closed, convex values, we apply the Convergence Theorem of Aubin and Cellina [1] to the inclusion (47). We obtain $z(t) \in \partial J(t, \bar{\gamma}(K(v + u_1)(t)))$ for a.e. $t \in (0, T)$, which implies $w \in \mathcal{N}_1 v$. The proof of the lemma is complete. □

Proof of Lemma 12. It is similar to the one of Lemma 9. We restrict ourselves to the proof of the property (c). So let $v \in \mathcal{V}$ and $w \in \mathcal{N}_1 v$. Hence $w(t) = \bar{\gamma}^* z(t)$ with $z(t) \in \partial J(t, \bar{\gamma}(v + u_1)(t))$ for a.e. $t \in (0, T)$. Using (2) we obtain

$$\|z(t)\|_{L^2(\Gamma;\mathbb{R}^N)} \leqslant c_1(1 + \beta\|\bar{\gamma}\| \, \|v(t) + u_1\|)$$

and subsequently

$$\begin{aligned}
\|w\|_{\mathcal{Z}^*} &\leqslant \|\bar{\gamma}^*\| \Big(\int_0^T \|z(t)\|^2_{L^2(\Gamma;\mathbb{R}^N)} \, \mathrm{d}t \Big)^{1/2} \\
&\leqslant c_1 \|\bar{\gamma}^*\| \Big(T + \beta^2 \|\bar{\gamma}\|^2 \int_0^T \|v(t) + u_1\|^2 \, \mathrm{d}t \Big)^2 \\
&\leqslant c_1 \|\bar{\gamma}^*\| \big(\sqrt{T} + \beta \|\bar{\gamma}\| \|v + u_1\|_{\mathcal{V}} \big).
\end{aligned}$$

Hence we have

$$|\langle\langle w, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}}| = |\langle\langle w, v \rangle\rangle_{\mathcal{Z}^* \times \mathcal{Z}}| \leqslant \beta \|w\|_{\mathcal{Z}^*} \|v\|_{\mathcal{V}} \leqslant c_1 \beta^2 \|\bar{\gamma}\|^2 \|v\|^2_{\mathcal{V}} + \tilde{c}\, \|v\|_{\mathcal{V}}$$

with $\tilde{c} > 0$, which implies $\langle\langle w, v \rangle\rangle_{\mathcal{V}^* \times \mathcal{V}} \geqslant -c_1 \beta^2 \|\bar{\gamma}\|^2 \|v\|^2_{\mathcal{V}} - \tilde{c}\, \|v\|_{\mathcal{V}}$. □

6. Applications

In order to apply the previous results, we will present in this section examples of dynamic viscoelastic contact problems which may be formulated in the form of Problems (I) and (II) and such that the hypotheses of Section 4 hold. Our existence theorems are also applicable to a general problem of viscoelastic masonry structures and to hemivariational inequalities arising in the theory of laminated viscoelastic Kirchhoff plates (see Naniewicz and Panagiotopoulos [28] and Panagiotopoulos [34]).

We consider a linear viscoelastic body occupying the domain Ω in $\mathbb{R}^N$ $(N=2,3)$ which is acted upon by volume forces and surface tractions and which may come in contact with a foundation on the part Γ_C of the boundary $\partial\Omega$. We are interested in the resulting dynamical process of the mechanical state of the body on the time interval $[0,T]$. We assume that the body is endowed with short memory (cf. Duvaut and Lions [9]) that is the state of the stress at the instant t depends only on the strain at that instant and at the immediately preceding instants. In this case (we use here the summation convention), we have

$$\sigma_{ij}(u)=b_{ijhk}\varepsilon_{kh}(u)+a_{ijhk}\frac{\partial}{\partial t}\varepsilon_{kh}(u), \quad i,j=1,\ldots,N, \tag{49}$$

where $u\colon \Omega\times(0,T)\to\mathbb{R}^N$ denotes the displacemrnt field, $\sigma=\sigma(u)$ is the stress tensor and the strain tensor $\varepsilon=\varepsilon(u)$ is given by $\varepsilon_{hk}(u)=\frac{1}{2}(u_{k,h}+u_{h,k})$. The viscosity coefficients a_{ijhk} and the elasticity coefficients b_{ijhk} satisfy the well known symmetry and ellipticity conditions. The dynamic behaviour of the body is described by the equilibrium equation

$$\sigma_{ij,j}(u)+f_i=u_i'' \quad \text{in } \Omega\times(0,T), \tag{50}$$

where f denotes the density of body force acting in $\Omega\times(0,T)$. We suppose that $\partial\Omega=\Gamma_D\cup\Gamma_N\cup\Gamma_C$ with meas $(\Gamma_D)>0$. The displacement $u=\{u_i\}$ and the tractions $F=\{F_i\}$ are prescribed on Γ_D and Γ_N, respectively, i.e.

$$\begin{cases} u_i=0 & \text{on } \Gamma_D\times(0,T)\\ S_i=F_i & \text{on } \Gamma_N\times(0,T)\end{cases} \tag{51}$$

where $S=\{S_i\}$, $S_i=\sigma_{ij}n_j$ denote the stress vector on Γ_N and $n=\{n_i\}$ is the outward unit normal to $\partial\Omega$. We assume that on a part Γ_C of the boundary a nonmonotone multivalued law holds between the displacement u and the reaction $-S$, that is

$$-S\in\partial j(u) \quad \text{on } \Gamma_C\times(0,T), \tag{52}$$

where j is a locally Lipschitz function on $\mathbb{R}^N$. Such boundary conditions appear in models for the dynamic adhesive contact problems (see Naniewicz and Panagiotopoulos [28], Panagiotopoulos [30, 34], Panagiotopoulos and Pop [35], Chau, Shillor and Sofonea [7] and the references therein, for adhesive contact effects, unilateral contact laws and nonmonotone frictional behaviour). In several classes of mechanical problems, similar boundary conditions may be defined between S and the velocity, i.e.

$$-S \in \partial j\left(\frac{\partial u}{\partial t}\right) \quad \text{on } \Gamma_C \times (0,T), \tag{53}$$

where again $j\colon \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function (we refer to Panagiotopoulos [33] and Goeleven, Miettinen and Panagiotopoulos [12] for general damped conditions and multivalued reaction–velocity laws). We mention that all nonconvex superpotential graphs in Chapter 4.6 of Naniewicz and Panagiotopoulos [28] (in particular the functions j defined as a minimum and as a maximum of quadratic convex functions) satisfy hypothesis H(j)(iii).

Multiplying the equilibrium equation by $v \in V$ and applying the Green theorem, we have

$$\langle u''(t), v\rangle_{V^*\times V} + a(u'(t), v) + b(u(t), v) - \int_{\Gamma_C} \sigma n \cdot v \, d\sigma(x) = \langle l, v\rangle_{V^*\times V}$$

for all $v \in V$ and a.e. $t \in (0,T)$, where $V = \{v \in H^1(\Omega;\mathbb{R}^N) : v_i = 0$ on $\Gamma_D\}$, $S = \sigma n$, $a(u,v) = \int_\Omega a_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)\,dx$, $b(u,v) = \int_\Omega b_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)\,dx$ and $\langle l, v\rangle = \int_\Omega f_i v_i\,dx + \int_{\Gamma_N} F_i v_i\,d\sigma(x)$. We suppose $f \in L^2(\Omega;\mathbb{R}^N)$ and $F \in L^2(\Gamma_N;\mathbb{R}^N)$. Together with the initial data, the variational formulation of the problem (49)–(52) reads as follows:

$$\begin{cases} \langle u''(t) + Au'(t) + Bu(t) - l, v\rangle_{V^*\times V} \\ \quad + \int_{\Gamma_C} j^0(\gamma u(t); \gamma v)\,d\sigma(x) \geqslant 0 \quad \text{for all } v \in V \text{ and a.e. } t \in (0,T) \\ u(0) = u_0, \ u'(0) = u_1, \end{cases} \tag{54}$$

where $\langle Au, v\rangle = a(u,v)$, $\langle Bu, v\rangle = b(u,v)$, $u_0 \in V$ and $u_1 \in H = L^2(\Omega;\mathbb{R}^N)$. Similarily, the formulation of the problem (49)–(51), (53) reads as follows:

$$\begin{cases} \langle u''(t) + Au'(t) + Bu(t) - l, v\rangle_{V^*\times V} \\ \quad + \int_{\Gamma_C} j^0(\gamma u'(t); \gamma v)\,d\sigma(x) \geqslant 0 \quad \text{for all } v \in V \quad \text{and a.e.} \quad t \in (0,T) \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{55}$$

Invoking Theorems 6 and 11 we obtain the existence of solutions to problems (54) and (55), respectively.

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